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Self-organized pulse generator in a reaction-diffusion system

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We carry out computer simulations of a Bonhoeffer–van der Pol-type reaction-diffusion equation to study the properties of propagating pulses and their collision. By choosing a suitable nonlinearity where a stable limit cycle solution coexists with an equilibrium uniform solution, it is shown that two pulses propagating to the opposite directions do not annihilate upon collision but generate a localized domain which persistently emits pulses traveling outward. The stability of the localized domain and the propagating pulses are explored numerically.

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Pattern formation and self-organization far from equilibrium have attracted much attention recently [1]. One of the features of systems far from equilibrium is a variety of curious dynamical orders which are not seen in thermal equilibrium. Reaction-diffusion equations have been used successfully for modeling these spatiotemporal structures. One example is the phase field approach [2,3] which is useful for computer simulations of melt growth. The Bonhoffer–van der Pol (BvP) -type equation which exhibits excitability has also been studied [4]. In fact, the BvP equation has been applied to various phenomena such as pulse propagation along the nerve axon, spiral waves in the Belousov-Zhabotinsky reaction, animal coating [5], and glow discharge [6–8]. Thus the BvP equation is one of the prototype model equations for pattern formation and self-organization far from equilibrium.

In this paper, we shall study the BvP equation in the parameter region where, if diffusion is absent, a uniform stationary state and a limit cycle oscillation coexist. We will show that when the diffusion turns on, propagating pulses exhibit quite unusual properties which have not been reported so far.

The BvP equation is the following coupled set of equations for the activator u and the inhibitor v:

$$\tau \frac{\partial u}{\partial t} = D_u \nabla^2 u + f(u) - v, \tag{1a}$$

$$\frac{\partial v}{\partial t} = D_v \nabla^2 v + u - \gamma v, \tag{1b}$$

where $f(u)$ contains cubiclike nonlinearity. Throughout this paper, we put $f(u)$ as

$$f(u) = \frac{1}{2}\left[\tanh\left(\frac{u-a}{\delta}\right) + \tanh\left(\frac{a}{\delta}\right)\right] - u.$$

If we choose $\delta = 0.05$ and $a = 0.15$, Eqs. (1) for $D_u = D_v = \gamma = 0$ and $\tau = 1$ admit both the stationary uniform solution $u = v = 0$ and a limit cycle oscillation, both of which are stable locally as is shown in Fig. 1. This coexistence

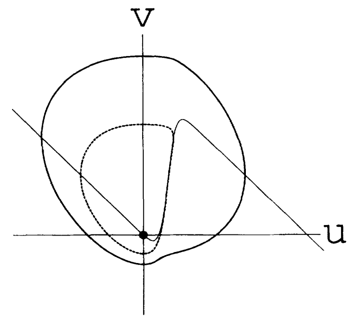

FIG. 1. Limit cycle orbit (thick line) and equilibrium solution (black circle) of Eqs. (1) for $\delta = 0.05$ and $a = 0.15$ in the u-v plane. The thin line indicates the function $v = f(u)$ whereas the dotted line means the separatrics.

appears in the interval $0.1 < a < 0.16$ for $\delta = 0.05$. When $a > 0.16$, a stable limit cycle oscillation does not exist and the system is simply excitable. To the authors' knowledge, the BvP equation (1) has not been studied in this parameter regime previously. Computer simulations shown below have been carried out for $D_u = \tau = 1$, $\gamma = 0$, and $\delta = 0.05$ and by changing other parameters D_v and a. The oscillatory property is strengthened when we decrease the value of a whereas a propagating pulse tends to be unstable when we increase the diffusion constant D_v of the inhibitor.

It should be noted that the parameters D_u and τ are of ordinary magnitude. This is quite in contrast to the previous studies [4,9–11] where these parameters are assumed to be extremely small so that we can apply a singular perturbation method. Smallness of the parameters a and δ in (1c) is also essential in the present problem. Coexistence of the uniform stationary state and a limit cycle oscillation emerges under these conditions. Furthermore the limit cycle oscillation in Fig. 1 still preserves the excitability in a sense that the motion in the u-v space is not uniform but slows down near $u = a$ and $v < 0$ and then accelerates for $u \geqslant a$ as if it is repelled from the state $u = v = 0$.

First we examine a head-on collision of two pulses in one dimension. It is well known that pulses in a dissipative sys-

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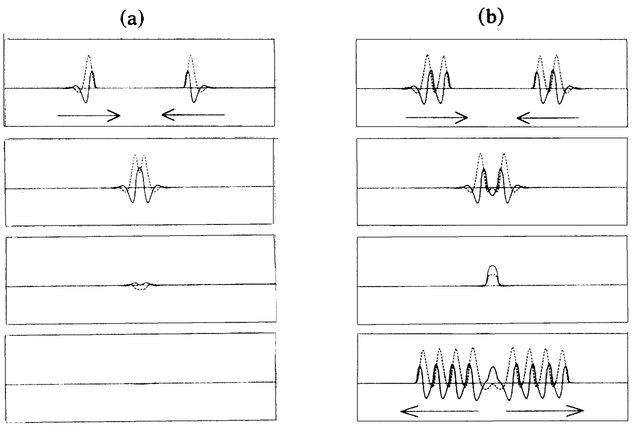

FIG. 2. (a) Collision and annihilation of two single pulses for $D_v=0$ and $a=0.15$. The full (dotted) line indicates the profile of u (v). The time steps are $t=0$, 10, 15, 20 from top to bottom. (b) Collision of pulse trains and formation of a pulse generator for $D_v=0$ and $a=0.15$. The time steps are $t=0$, 20, 30, 60 from top to bottom.

tem generally annihilate upon collision. This is indeed the case in Eqs. (1) for sufficiently large values of a where the system is excitable. We have found, however, that a qualitatively different behavior occurs in the coexistence region. Two single pulses decay upon collision as shown in Fig. 2(a) where $D_v=0$ and $a=0.15$. This should be compared with a collision of two-wave trains in Fig. 2(b) for the same values of the parameters. In this process, the front trains annihilate as usual but a localized oscillatory domain forms after the collision of the second trains and furthermore this domain produces sustained wave trains propagating outward. We have also simulated collision of an n-wave train and an m-wave train with $n>m>2$. In this case, an $(n-m)$-wave train survives and a similar localized domain which emits propagating waves is constituted. We call the localized domain a self-organized pulse generator. It is noted here that the oscillating amplitude of u and v and the period at the center of the domain are almost identical to those of the limit cycle without diffusion.

One can see from Fig. 2(b) that the region of the pulse generator gradually expands with time as if the oscillating domain invades the surrounding quiescent state while emitting the outgoing waves. We have found for longer runs that the speed and the spatial period of the wave train gradually increase with time while the domain itself expands. This slow change of wave trains is attributed to a phase diffusion. However, the intrinsic asymptotic behavior is quite difficult to analyze numerically because of the boundary effect as we have imposed the Neumann boundary condition at the system boundary.

In order to examine the stability of the pulse generator and the emitted wave trains we have carried out alternative simulations. That is, we start with the initial condition $u(x,0)=\exp(-x^2)$ and $v(x,0)=0$ for $-L<x<L$ with $L=100$. We have found that this initial inhomogenity triggers a pulse generator and wave trains as in Fig. 2(b) after collision.

We have explored the time evolution of the pulse generator and the emitted waves by changing the parameters D_v

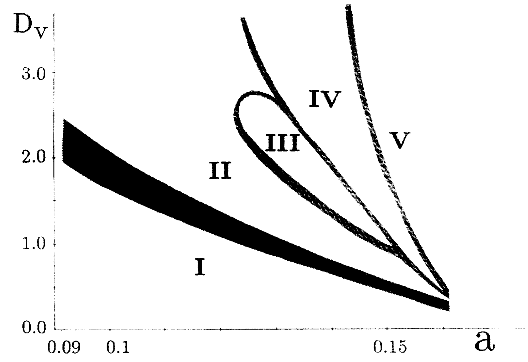

FIG. 3. Stability diagram in D_v-a plane.

and a. Figure 3 summarizes the results. Typically four different spatiotemporal patterns are classified as is indicated respectively by I, II, . . . ,IV. When the diffusion constant D_v is extremely large as in the region V, the initial localized domain decays and disappears.

In region I where D_v is small, the initial localized domain undergoes oscillation and it forms a pulse generator which produces successive wave trains. This is identical to the situation after collision shown in Fig. 2(b). Figure 4(a) shows the spatiotemporal pattern where the contour $u=0.001$ is plotted.

In region II, the initial localized domain still acts as a pulse generator. However, the propagating pulse trains become unstable. That is, an emitted pulse propagates for a while and then dies out. This disappearance of a pulse occurs repeatedly at the front of a pulse train. Since the speed of a

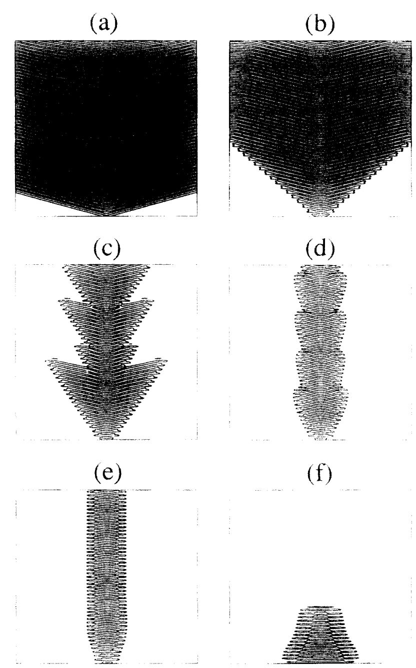

FIG. 4. Spatiotemporal patterns of the pulse generator and the emitted pulses for (a) $D_v=0$, (b) 0.7, (c) 0.9, (d) 1.1, (e) 1.6 and (f) 6.0. The parameter a is fixed as $a=0.15$. The lines indicate the contour lines of $u=0.001$. The abscissa is the space axis whereas the ordinate is the time axis.

pulse train is larger than the decay rate of a front pulse, the front moves slowly outward. Figure 4(b) indicates the behavior for $D_v=0.7$ and $a=0.15$. It is also noted that decay of a pulse does not necessarily occur only at the front of a pulse train. It is often observed in the middle of a pulse train as shown in Fig. 4(c) for $D_v=0.9$. This can be understood as follows; Since the pulse generator is expanding, a situation such that two pulses emitted successively whose distance is too short occurs for intermediate values of D_v in region II. In this case, one of the pulses cannot survive and disappears. For larger values of D_v where expansion of the pulse generator becomes negligible, only the front pulse decays.

The zig-zag pattern in Fig. 4(c) becomes smooth in region III. Figure 4(d) shows an example for $D_v=1.1$ and $a=0.15$. What happens is an oscillation of the domain width as well as inside of the domain. We emphasize that this oscillation should not be confused with another type of oscillation of a domain in the BvP equation (1). It has been reported that a stable localized motionless domain, which is a solution of Eqs. (1) for small values of D_u, begins to oscillate when we decrease τ. This was called a breathing motion [10–12]. Recall that the above-mentioned more complicated "double" oscillation observed in region III emerges for D_u and τ of the order of unity.

Although not displayed in the figures, the double oscillation appears after a long transient in region III with fairly large values of D_v such as $D_v=2.0$ and $a=0.13$. In the transient regime, the pattern looks like that in Fig. 4(c).

When we further increase D_v, we enter region IV where emitted pulses cannot propagate any more. This is because the inhibitor generated by the reaction term in Eq. (1a) rapidly diffuses to the front of a pulse so that propagation is inhibited. However, the domain is still oscillating. An example for $D_v=1.6$ and $a=0.15$ is shown in Fig. 4(e). This is somehow similar to the breather solution well known in the nonlinear Schrödinger equation although the present system (1) is purely dissipative. By a detailed numerical analysis, we have confirmed that the change between in Figs. 4(d) and 4(e) occurs as a supercritical Hopf bifurcation.

In region V, the uniform state becomes more stable. Figure 4(f) shows that an oscillatory domain with the initial width $l=100$ shrinks and eventually disappears for $D_v=6.0$ and $a=0.15$.

The persistent outgoing wave train corresponds to a concentric wave (target pattern) in higher dimensions. We have indeed verified numerically that a target pattern emerges from a self-organized pulse generator localized in two dimensions as shown in Fig. 5. It is emphasized that the target pattern appears without any heterogeneous pacemaker. What is necessary is only an initial concentration deviation. We have also obtained spiral waves and the reconnection of pulses in two dimensions. However, these results will be reported separately elsewhere.

There are several other model equations which have a target pattern without heterogeneous pacemakers. However we emphasize that there are some essential differences. First of all, BvP equation (1) is a two-variable model whereas almost all of the previous models are three– (or more) variable. For instance, a spatially localized concentric wave solution has been obtained in the complex time-dependent Ginzburg-Landau (TDGL) equation coupled with a phase

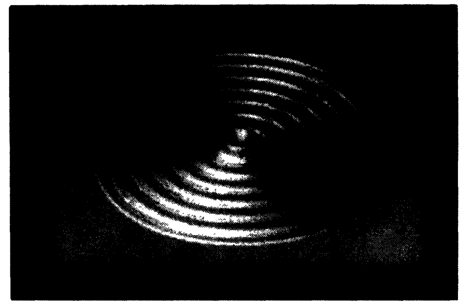

FIG. 5. Two-dimensional target pattern for $D_v=0$ and $a=0.15$. The system size is 200×200.

variable or with another complex field [13]. Computer simulations of a BvP-type model equation for glow discharge [6] have shown automatically excited pulse trains similar to that in Fig. 4(a). However, the time-evolution equation contains a long range nonlocal interaction. In order to make this interaction short ranged like a diffusion term, one needs to introduce an extra variable so that the model in Ref. [6] is essentially three-variable. Mikhailov [14] has also proposed a three-variable model for a nonlocalized target pattern. Thus the mechanism of the pulse generation in these model systems is expected to be different from that of Eqs. (1).

A two-variable equation closely related to the present system is probably the one studied by Thual and Fauve [15]. They have carried out simulations of the complex TDGL equation with a subcritical Hopf bifurcation and found a localized target solution. Since their system has a coexistence of a uniform solution and a limit cycle, the basic character is quite similar to Eqs. (1). However, it should be noted that in order for a stable wave train to exist, the excitability is essential. The complex TDGL equation is not excitable but simply oscillatory and therefore does not admit a stable wave train.

By computer simulations of a generalized complex TDGL equation, Brant and Deissler [16] have found that pulses in a dissipative system do not necessarily disappear upon collision. In their results, a collision of two pulses is similar to that of two solitons in an integrable system. As shown above, Eqs. (1) exhibit more variety of spatiotemporal patterns in a collision.

In summary, we have investigated the pattern dynamics of Eqs. (1). One of the most important properties of the system is the existence of a locally stable limit cycle which contains, to some extent, an excitability character. A pulse generator is self-organized by a collision of pulses or by a local concentration inhomogeneity. In two dimensions, this causes a target pattern which is apparently similar to that observed in the Belousov-Zhabotinsky reaction. In the present results, the frequency of oscillation at the center of a pulse generator is close to that of a uniform oscillation, which is not in accord with experiments [17]. However a detailed comparison is meaningless since the parameters D_u and τ are chosen to be of order unity in the present system. Thus, at present, there are no real experiments which support directly the predictions given above. However some aspect of the results is of great importance in its own right. For example, a collision of single pulses does not cause a pulse generator but that of two (or more) wave trains does. This means that emission of waves depends on the number of stimuli. We of course note the implication of this to a neural system.

A global phase diagram is obtained in the D_v-a plane. The basic feature is the coexistence of the limit cycle and a uniform state. When D_v and a are small, a domain of limit cycle oscillation invades the surrounding uniform state. By increasing the value of D_v, a propagating pulse tends to be unstable so that the expansion of domain is suppressed. In this way, the phase diagram can be interpreted qualitatively. However a quantitative analytical theory seems quite difficult, which is left for a future study.

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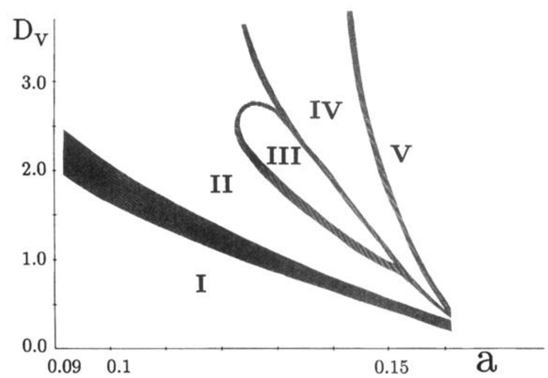

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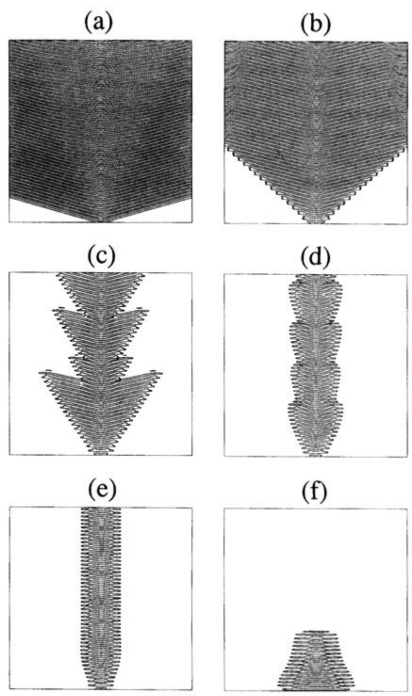

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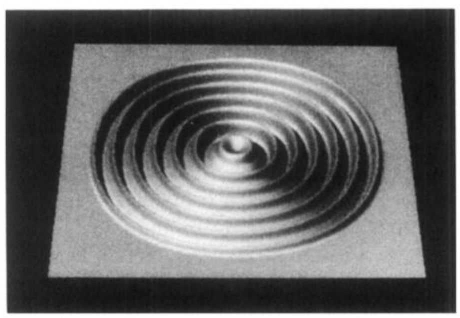

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